

# CROFTON FORMULAE FOR TENSORIAL CURVATURE MEASURES: THE GENERAL CASE

DANIEL HUG AND JAN A. WEIS

**ABSTRACT.** The tensorial curvature measures are tensor-valued generalizations of the curvature measures of convex bodies. In a previous work, we obtained kinematic formulae for all tensorial curvature measures. As a consequence of these results, we now derive a complete system of Crofton formulae for such tensorial curvature measures on convex bodies and for their (nonsmooth) generalizations on convex polytopes. These formulae express the integral mean of the tensorial curvature measures of the intersection of a given convex body (resp. polytope) with a uniform affine  $k$ -flat in terms of linear combinations of tensorial curvature measures of the given convex body (resp. polytope). The considered tensorial curvature measures generalize those studied formerly in the context of Crofton-type formulae, and the coefficients involved in these results are substantially less technical and structurally more transparent than in previous works.

## 1. INTRODUCTION

The *classical Crofton formula* is a major result in integral geometry. It expresses the integral mean of the intrinsic volume of a convex body intersected with a uniform affine subspace of the underlying Euclidean space in terms of another intrinsic volume of this convex body. More precisely, for a *convex body*  $K \in \mathcal{K}^n$  (a nonempty, compact, convex set) in the  $n$ -dimensional Euclidean space  $\mathbb{R}^n$ ,  $n \in \mathbb{N}$ , the classical Crofton formula (see [23, (4.59)]) states that

$$(1) \quad \int_{A(n,k)} V_j(K \cap E) \mu_k(dE) = \alpha_{njk} V_{n-k+j}(K),$$

for  $k \in \{0, \dots, n\}$  and  $j \in \{0, \dots, k\}$ , where  $A(n, k)$  is the affine Grassmannian of  $k$ -flats in  $\mathbb{R}^n$ , on which  $\mu_k$  denotes the motion invariant Haar measure, normalized as in [24, p. 588], and  $\alpha_{njk} > 0$  is an explicitly known constant (see [23, Theorem 4.4.2]).

The functionals  $V_i : \mathcal{K}^n \rightarrow \mathbb{R}$ , for  $i \in \{0, \dots, n\}$ , appearing in (1), are the *intrinsic volumes*, which occur as the coefficients of the monomials in the *Steiner formula*

$$(2) \quad \mathcal{H}^n(K + \epsilon B^n) = \sum_{j=0}^n \kappa_{n-j} V_j(K) \epsilon^{n-j},$$

which holds for all convex bodies  $K \in \mathcal{K}^n$  and  $\epsilon \geq 0$ . Here,  $\mathcal{H}^n$  is the  $n$ -dimensional Hausdorff measure (Lebesgue measure, volume),  $+$  denotes the Minkowski addition in  $\mathbb{R}^n$ , and  $\kappa_n$  is the volume of the Euclidean unit ball  $B^n$  in  $\mathbb{R}^n$ . Properties of the intrinsic volume  $V_i$  such as continuity, isometry invariance, homogeneity, and additivity (valuation property)

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are derived from corresponding properties of the volume functional. A key result for the intrinsic volumes is *Hadwiger's characterization theorem* (see [7, 2. Satz]), which states that  $V_0, \dots, V_n$  form a basis of the vector space of continuous and isometry invariant real-valued valuations on  $\mathcal{K}^n$ . One of its numerous applications is a concise proof of (1).

A natural way to extend the classical Crofton formula is to apply the integration over the affine Grassmannian  $A(n, k)$  to functionals which generalize the intrinsic volumes. One of these generalizations concerns tensor-valued valuations on  $\mathcal{K}^n$ . Their systematic investigation started with a characterization theorem, similar to the aforementioned result due to Hadwiger. However, integral geometric formulae, including a Crofton formula, for *quermassvectors* (vector-valued generalizations of the intrinsic volumes) have already been found by Hadwiger & Schneider and Schneider, in 1971/72 (see [8, 18, 19]). More recently in 1997, McMullen generalized these vector-valued valuations even further, and introduced tensor-valued generalizations of the intrinsic volumes (see [17]). Only two years later Alesker generalized Hadwiger's characterization theorem (see [1, Theorem 2.2]) by showing that the vector space of continuous and isometry covariant tensor-valued valuations on  $\mathcal{K}^n$  is spanned by the tensor-valued versions of the intrinsic volumes, the *Minkowski tensors*, multiplied with suitable powers of the metric tensor in  $\mathbb{R}^n$ . However, these valuations are not linearly independent, as shown by McMullen (see [17]) and further investigated by Hug, Schneider & Schuster (see [13]). This is one reason why an approach to explicit integral geometric formulae via characterization theorems does not seem to be technically feasible. Nevertheless, great progress in this direction has been made by different methods. In 2008, Hug, Schneider & Schuster proved a set of Crofton formulae for the Minkowski tensors (see [12, Theorem 2.1–2.6]). A totally different algebraic approach has been developed by Bernig & Hug to obtain various integral geometric formulae for the translation invariant Minkowski tensors (see [3]).

On the other hand, localizations of the intrinsic volumes yield other types of generalizations. The *support measures* are weakly continuous, locally defined and motion equivariant valuations on convex bodies with values in the space of finite measures on Borel subsets of  $\mathbb{R}^n \times \mathbb{S}^{n-1}$ , where  $\mathbb{S}^{n-1}$  denotes the Euclidean unit sphere in  $\mathbb{R}^n$ . They are determined by a local version of (2), and form a crucial example of localizations of the intrinsic volumes, which are simply the total support measures. Furthermore, their marginal measures on Borel subsets of  $\mathbb{R}^n$  are called *curvature measures*, and the ones on Borel subsets of  $\mathbb{S}^{n-1}$  are called *surface area measures*. For the area and curvature measures, Schneider showed characterization theorems (see [20, 21]), similar to the one due to Hadwiger in the global case. It took some time until in 1995, Glasauer proved a characterization theorem for the curvature measures, even without the need of requesting the valuation property (see [5, Satz 4.2.1]). As to integral geometry, in 1959 Federer [4] proved Crofton formulae for curvature measures, even in the more general setting of sets with positive reach. Certain Crofton formulae for support measures were proved by Glasauer in 1997 (see [6, Theorem 3.2]). However, his results require a special set operation on support elements of the involved convex bodies and affine subspaces.

Interestingly, the combination of Minkowski tensors and localizations leads to a better understanding of integral geometric formulae. In recent years, Schneider introduced *local tensor valuations* (see [22]), which were then further studied by Hug & Schneider (see [9, 10, 11]). They introduced particular tensor-valued support measures, the *local Minkowski tensors* on convex bodies (and nonsmooth generalizations on polytopes), which (as their

name suggests) can also be seen as localizations of the Minkowski tensors. They proved several different characterization results for these (generalized) local Minkowski tensors in the just mentioned works. This led us to consider their marginal measures on Borel subsets of  $\mathbb{R}^n$ , the *tensorial curvature measures* and their (nonsmooth) generalizations on convex polytopes. Preceding this work, the present authors derived a set of Crofton formulae for a different version of these tensorial curvature measures, defined with respect to the (random) intersecting affine subspace, and as a consequence of these results also obtained Crofton formulae for some of the (original) tensorial curvature measures (see [14]). As a far reaching generalization of previous results, a complete set of kinematic formulae for the (generalized) tensorial curvature measures has been proved in [15].

The aim of the present work is to prove a complete set of Crofton formulae for the generalized tensorial curvature measures. This complements the particular results for extrinsic tensorial curvature measures and Minkowski tensors obtained in [14] and [25]. The current approach is basically an application of the kinematic formulae for generalized tensorial curvature measures derived in [15]. The connection between local kinematic and local Crofton formulae is well known for the scalar curvature measures. There it is used to determine the coefficients in the kinematic formulae. The basic strategy there is as follows. First, the kinematic formulae are proved, but the involved coefficients remain undetermined, since the required direct calculation seemed to be infeasible. Then Crofton formulae are derived which involve the same constants. In the latter, the determination of the coefficients turns out to be an easy task, which is accomplished by evaluating the result for balls of different radii. In the tensorial framework, this approach breaks down, since the explicit calculation of integral mean values of tensorial curvature measures for sufficiently many examples (template method) does not seem to be possible. Instead, the required coefficients were determined by a direct derivation in the kinematic formulae for tensorial curvature measures [15], and from this we now can derive explicit Crofton formulae, otherwise following the preceding reasoning. We also refer to [15] for a more thorough discussion of related work.

The present contribution is structured as follows. In Section 2, we fix our notation and collect various auxiliary results which will be needed. Section 3 contains the main results. First, we state the Crofton formulae for the (nonsmooth) generalized tensorial curvature measures on the space  $\mathcal{P}^n$  of convex polytopes in  $\mathbb{R}^n$ . Then we provide the formulae for all the tensorial curvature measures, for which a smooth extension to  $\mathcal{K}^n$  exists. Finally, we recover some (partly known) special cases. In Section 4, we first recapitulate the kinematic formulae, in order to apply these in the proofs of the main results and the corollaries. In the final Section 5 we show that the tensorial curvature measures are essentially all linearly independent.

## 2. PRELIMINARIES

We work in the  $n$ -dimensional Euclidean space  $\mathbb{R}^n$ , equipped with its usual topology generated by the standard scalar product  $\langle \cdot, \cdot \rangle$  and the corresponding Euclidean norm  $\| \cdot \|$ . For a topological space  $X$ , we denote the Borel  $\sigma$ -algebra on  $X$  by  $\mathcal{B}(X)$ .

We denote the rotation group on  $\mathbb{R}^n$  by  $\mathrm{SO}(n)$ , and we write  $\nu$  for the Haar probability measure on  $\mathrm{SO}(n)$ . By  $G(n, k)$  (resp.  $A(n, k)$ ), for  $k \in \{0, \dots, n\}$ , we denote the Grassmannian of  $k$ -dimensional linear (resp. affine) subspaces of  $\mathbb{R}^n$ . We write  $\mu_k$  for the rotation invariant Haar probability measure on  $A(n, k)$ . The directional space of an affine  $k$ -flat  $E \in A(n, k)$  is denoted by  $E^0 \in G(n, k)$  and its orthogonal complement by

$E^\perp \in G(n, n-k)$ . The orthogonal projection of a vector  $x \in \mathbb{R}^n$  to a linear subspace  $L$  of  $\mathbb{R}^n$  is denoted by  $p_L(x)$ .

The vector space of symmetric tensors of rank  $p \in \mathbb{N}_0$  over  $\mathbb{R}^n$  is denoted by  $\mathbb{T}^p$ , and the corresponding algebra of symmetric tensors over  $\mathbb{R}^n$  by  $\mathbb{T}$ . The symmetric tensor product of two tensors  $T, U \in \mathbb{T}$  is denoted by  $TU$ , and for  $q \in \mathbb{N}_0$  and a tensor  $T \in \mathbb{T}$  we write  $T^q$  for the  $q$ -fold tensor product; see also the introductory chapters of [16] for further details and references. Identifying  $\mathbb{R}^n$  with its dual space via its scalar product, we interpret a symmetric tensor of rank  $p$  as a symmetric  $p$ -linear map from  $(\mathbb{R}^n)^p$  to  $\mathbb{R}$ . One special tensor is the *metric tensor*  $Q \in \mathbb{T}^2$ , defined by  $Q(x, y) := \langle x, y \rangle$  for  $x, y \in \mathbb{R}^n$ . For an affine  $k$ -flat  $E \in A(n, k)$ ,  $k \in \{0, \dots, n\}$ , the metric tensor  $Q(E)$  associated with  $E$  is defined by  $Q(E)(x, y) := \langle p_{E^0}(x), p_{E^0}(y) \rangle$  for  $x, y \in \mathbb{R}^n$ .

We define the tensorial curvature measures using their relation to the support measures, which we therefore introduce first. For a convex body  $K \in \mathcal{K}^n$  and  $x \in \mathbb{R}^n$ , we denote the metric projection of  $x$  onto  $K$  by  $p(K, x)$ , and for  $x \in \mathbb{R}^n \setminus K$  we define  $u(K, x) := (x - p(K, x)) / \|x - p(K, x)\|$ . For  $\epsilon > 0$  and a Borel set  $\eta \subset \Sigma^n := \mathbb{R}^n \times \mathbb{S}^{n-1}$ ,

$$M_\epsilon(K, \eta) := \{x \in (K + \epsilon B^n) \setminus K : (p(K, x), u(K, x)) \in \eta\}$$

is a local parallel set of  $K$  which satisfies the *local Steiner formula*

$$(3) \quad \mathcal{H}^n(M_\epsilon(K, \eta)) = \sum_{j=0}^{n-1} \kappa_{n-j} \Lambda_j(K, \eta) \epsilon^{n-j}, \quad \epsilon \geq 0.$$

This relation determines the *support measures*  $\Lambda_0(K, \cdot), \dots, \Lambda_{n-1}(K, \cdot)$  of  $K$ , which are finite Borel measures on  $\mathcal{B}(\Sigma^n)$ . Obviously, a comparison of (3) and the Steiner formula (2) yields  $V_j(K) = \Lambda_j(K, \Sigma^n)$ . Further information on these measures and functionals can be found in [23, Chap. 4.2].

Let  $\mathcal{P}^n \subset \mathcal{K}^n$  denote the space of convex polytopes in  $\mathbb{R}^n$ . For a polytope  $P \in \mathcal{P}^n$  and  $j \in \{0, \dots, n\}$ , we denote the set of  $j$ -dimensional faces of  $P$  by  $\mathcal{F}_j(P)$  and the normal cone of  $P$  at a face  $F \in \mathcal{F}_j(P)$  by  $N(P, F)$ . Then, the  $j$ th support measure  $\Lambda_j(P, \cdot)$  of  $P$  is explicitly given by

$$\Lambda_j(P, \eta) = \frac{1}{\omega_{n-j}} \sum_{F \in \mathcal{F}_j(P)} \int_F \int_{N(P, F) \cap \mathbb{S}^{n-1}} \mathbb{1}_\eta(x, u) \mathcal{H}^{n-j-1}(du) \mathcal{H}^j(dx)$$

for  $\eta \in \mathcal{B}(\Sigma^n)$  and  $j \in \{0, \dots, n-1\}$ , where  $\mathcal{H}^j$  denotes the  $j$ -dimensional Hausdorff measure and  $\omega_n$  is the  $(n-1)$ -dimensional volume (Hausdorff measure) of  $\mathbb{S}^{n-1}$ .

For a polytope  $P \in \mathcal{P}^n$ , we define the *generalized tensorial curvature measure*

$$\phi_j^{r,s,l}(P, \cdot), \quad j \in \{0, \dots, n-1\}, \quad r, s, l \in \mathbb{N}_0,$$

as the Borel measure on  $\mathcal{B}(\mathbb{R}^n)$  which is given by

$$\phi_j^{r,s,l}(P, \beta) := c_{n,j}^{r,s,l} \frac{1}{\omega_{n-j}} \sum_{F \in \mathcal{F}_j(P)} Q(F)^l \int_{F \cap \beta} x^r \mathcal{H}^j(dx) \int_{N(P, F) \cap \mathbb{S}^{n-1}} u^s \mathcal{H}^{n-j-1}(du),$$

for  $\beta \in \mathcal{B}(\mathbb{R}^n)$ , where

$$c_{n,j}^{r,s,l} := \frac{1}{r!s!} \frac{\omega_{n-j}}{\omega_{n-j+s}} \frac{\omega_{j+2l}}{\omega_j} \text{ if } j \neq 0, \quad c_{n,0}^{r,s,0} := \frac{1}{r!s!} \frac{\omega_n}{\omega_{n+s}}, \quad \text{and } c_{n,0}^{r,s,l} := 1 \text{ for } l \geq 1.$$

Note that if  $j = 0$  and  $l \geq 1$ , then we have  $\phi_0^{r,s,l} \equiv 0$ . In all other cases the factor  $1/\omega_{n-j}$  in the definition of  $\phi_j^{r,s,l}(P, \beta)$  and the factor  $\omega_{n-j}$  involved in the constant  $c_{n,j}^{r,s,l}$  cancel.

For a general convex body  $K \in \mathcal{K}^n$ , we define the *tensorial curvature measure*

$$\phi_n^{r,0,l}(K, \cdot), \quad r, l \in \mathbb{N}_0,$$

as the Borel measure on  $\mathcal{B}(\mathbb{R}^n)$  which is given by

$$\phi_n^{r,0,l}(K, \beta) := c_{n,n}^{r,0,l} Q^l \int_{K \cap \beta} x^r \mathcal{H}^n(dx),$$

for  $\beta \in \mathcal{B}(\mathbb{R}^n)$ , where  $c_{n,n}^{r,0,l} := \frac{1}{r!} \frac{\omega_{n+2l}}{\omega_n}$ . For the sake of convenience, we extend these definitions by  $\phi_j^{r,s,0} := 0$  for  $j \notin \{0, \dots, n\}$  or  $r \notin \mathbb{N}_0$  or  $s \notin \mathbb{N}_0$  or  $j = n$  and  $s \neq 0$ . Finally, we observe that for  $P \in \mathcal{P}^n$ ,  $r = s = l = 0$ , and  $j = 0, \dots, n-1$ , the scalar-valued measures  $\phi_j^{0,0,0}(P, \cdot)$  are just the curvature measures  $\phi_j(P, \cdot)$ , that is, the marginal measures on  $\mathbb{R}^n$  of the support measures  $\Lambda_j(P, \cdot)$ , which therefore can be extended from polytopes to general convex bodies, and  $\phi_n^{0,0,0}(K, \cdot)$  is the restriction of the  $n$ -dimensional Hausdorff measure to  $K \in \mathcal{K}^n$ .

We emphasize that in the present work,  $\phi_j^{r,s,l}(P, \cdot)$  and  $\phi_n^{r,0,l}(K, \cdot)$  are Borel measures on  $\mathbb{R}^n$  and not on  $\mathbb{R}^n \times \mathbb{S}^{n-1}$ , as in [9], and also the normalization is slightly adjusted as compared to [9] (where the normalization was not a relevant issue). However, we stick to the definition and normalization of our preceding work [15], where the connection to the generalized local Minkowski tensors  $\tilde{\phi}_j^{r,s,l}$  is described and where also the properties and characterization results for these measures are discussed in more detail.

It has been shown in [9] that the generalized local Minkowski tensor  $\tilde{\phi}_j^{r,s,l}$  has a continuous extension to  $\mathcal{K}^n$  which preserves all other properties if and only if  $l \in \{0, 1\}$ ; see [9, Theorem 2.3] for a stronger characterization result. Globalizing any such continuous extension in the  $\mathbb{S}^{n-1}$ -coordinate, we obtain a continuous extension for the generalized tensorial curvature measures. For  $l = 0$ , this extension can be easily expressed via the support measures. We call these the *tensorial curvature measures*. For a convex body  $K \in \mathcal{K}^n$ , a Borel set  $\beta \in \mathcal{B}(\mathbb{R}^n)$ , and  $r, s \in \mathbb{N}_0$ , they are given by

$$(4) \quad \phi_j^{r,s,0}(K, \beta) := c_{n,j}^{r,s,0} \int_{\beta \times \mathbb{S}^{n-1}} x^r u^s \Lambda_j(K, d(x, u)),$$

for  $j \in \{0, \dots, n-1\}$ , whereas  $\phi_n^{r,0,l}(K, \beta)$  has already been defined for all  $K \in \mathcal{K}^n$ .

The valuations  $Q^m \phi_j^{r,s,l}$  on  $\mathcal{P}^n$  are linearly independent, where  $m, j, r, s, l \in \mathbb{N}_0$ ,  $j \in \{0, \dots, n\}$  with  $l = 0$ , if  $j \in \{0, n-1\}$ , and  $s = l = 0$ , if  $j = n$ . This can be shown in essentially the same way as it was done for the local Minkowski tensors in [9, Theorem 3.1].

The statements of our results involve the classical *Gamma function*. For all  $z \in \mathbb{C} \setminus \{0, -1, \dots\}$  (see [2, (2.7)]), it can be defined via the Gaussian product formula

$$\Gamma(z) := \lim_{a \rightarrow \infty} \frac{a^z a!}{z(z+1) \cdots (z+a)}.$$

This definition implies, for  $c \in \mathbb{R} \setminus \mathbb{Z}$  and  $m \in \mathbb{N}_0$ , that

$$(5) \quad \frac{\Gamma(-c+m)}{\Gamma(-c)} = (-1)^m \frac{\Gamma(c+1)}{\Gamma(c-m+1)}.$$

The Gamma function has simple poles at the nonpositive integers. The right side of relation (5) provides a continuation of the left side at  $c \in \mathbb{N}_0$ , where  $\Gamma(c-m+1)^{-1} = 0$  for  $c < m$ .

### 3. THE CROFTON FORMULAE

In this work, we establish a complete set of Crofton formulae for the tensorial curvature measures of polytopes. That is, for  $P \in \mathcal{P}^n$  and  $\beta \in \mathcal{B}(\mathbb{R}^n)$ , we explicitly express integrals of the form

$$\int_{A(n,k)} \phi_j^{r,s,l}(P \cap E, \beta \cap E) \mu_k(dE)$$

in terms of generalized tensorial curvature measures of  $P$ , evaluated at  $\beta$ . Furthermore, for  $l = 0, 1$ , the tensorial measures are defined on  $\mathcal{K}^n \times \mathcal{B}(\mathbb{R}^n)$ , and therefore we also consider the Crofton integrals

$$\int_{A(n,k)} \phi_j^{r,s,l}(K \cap E, \beta \cap E) \mu_k(dE)$$

for  $K \in \mathcal{K}^n$ ,  $\beta \in \mathcal{B}(\mathbb{R}^n)$ ,  $l = 0, 1$ .

All results which are stated in the following, extend by additivity to finite unions of polytopes or convex bodies.

**3.1. Tensorial curvature measures on polytopes.** First, we separately state a formula for  $j = k$ .

**Theorem 1.** *Let  $P \in \mathcal{P}^n$ ,  $\beta \in \mathcal{B}(\mathbb{R}^n)$ , and  $k, r, s, l \in \mathbb{N}_0$  with  $k \leq n$ . Then,*

$$\int_{A(n,k)} \phi_k^{r,s,l}(P \cap E, \beta \cap E) \mu_k(dE) = \mathbb{1}\{s \text{ even}\} \frac{1}{(2\pi)^s \left(\frac{s}{2}\right)!} \frac{\Gamma(\frac{n-k+s}{2})}{\Gamma(\frac{n-k}{2})} \phi_n^{r,0,\frac{s}{2}+l}(P, \beta).$$

Theorem 1 generalizes Theorem 2.1 in [12]. In fact, setting  $l = 0$  and  $\beta = \mathbb{R}^n$  one obtains the known result for Minkowski tensors. If  $l \in \{0, 1\}$ , one can even formulate Theorem 1 for a convex body, as in both of these cases all appearing valuations are defined on  $\mathcal{K}^n$ . For  $k = n$ , the integral on the left-hand side of the formula in Theorem 1 is trivial. However, note that on the right-hand side the quotient of the Gamma functions basically equals  $\mathbb{1}\{s = 0\}$ .

Next, we state the formula for general  $j < k$ .

**Theorem 2.** *Let  $P \in \mathcal{P}^n$ ,  $\beta \in \mathcal{B}(\mathbb{R}^n)$ , and  $j, k, r, s, l \in \mathbb{N}_0$  with  $j < k \leq n$ , and with  $l = 0$  if  $j = 0$ . Then,*

$$\int_{A(n,k)} \phi_j^{r,s,l}(P \cap E, \beta \cap E) \mu_k(dE) = \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{i=0}^m d_{n,j,k}^{s,l,i,m} Q^{m-i} \phi_{n-k+j}^{r,s-2m,l+i}(P, \beta),$$

where

$$\begin{aligned} d_{n,j,k}^{s,l,i,m} &:= \frac{(-1)^i}{(4\pi)^m m!} \frac{\binom{m}{i}}{\pi^i} \frac{(i+l-2)!}{(l-2)!} \frac{\Gamma(\frac{n-k+j+1}{2}) \Gamma(\frac{k+1}{2})}{\Gamma(\frac{n+1}{2}) \Gamma(\frac{j+1}{2})} \\ &\quad \times \frac{\Gamma(\frac{n-k+j}{2} + 1)}{\Gamma(\frac{n-k+j+s}{2} + 1)} \frac{\Gamma(\frac{j+s}{2} - m + 1)}{\Gamma(\frac{j}{2} + 1)} \frac{\Gamma(\frac{n-k}{2} + m)}{\Gamma(\frac{n-k}{2})}. \end{aligned}$$

For  $k = n$  the coefficient in Theorem 2 is simply given by

$$d_{n,j,n}^{s,l,i,m} = \mathbb{1}\{i = m = 0\}$$

so that the result is a tautology.

Several remarkable facts concerning the coefficients  $d_{n,j,k}^{s,l,i,m}$  should be recalled from [15]. First, the ratio  $(i+l-2)!/(l-2)!$  has to be interpreted in terms of Gamma functions and

relation (5) if  $l \in \{0, 1\}$ . The corresponding special cases will be considered separately in the following two theorems and the subsequent corollaries. Second, the coefficients are independent of the tensorial parameter  $r$  and depend only on  $l$  through the ratio  $(i + l - 2)!/(l - 2)!$ . Third, only tensors  $\phi_{n-k+j}^{r,s-2m,p}(P, \beta)$  with  $p \geq l$  show up on the right side of the kinematic formula. Using Legendre's duplication formula, we could shorten the given expressions for the coefficients  $d_{n,j,k}^{s,l,i,m}$  even further. However, the present form has the advantage of exhibiting that the factors in the second line cancel each other if  $s = 0$  (and hence also  $m = i = 0$ ). Furthermore, the coefficients are signed in contrast to the classical kinematic formula. We shall see below that for  $l \in \{0, 1\}$  all coefficients are nonnegative.

**3.2. Tensorial curvature measures on convex bodies.** The generalized tensorial curvature measures  $\phi_j^{r,s,l}$  can be continuously extended to all convex bodies if  $l \in \{0, 1\}$ . In these two cases, Theorem 1 holds for general convex bodies as well. For this reason, we restrict our attention to the cases where  $j < k$  in the following. The next theorems are stated without a proof, as they basically follow from Theorem 2 and approximation of the given convex body by polytopes (using the weak continuity of the curvature measures and the usual arguments needed to take care of exceptional positions).

We start with the formula for  $l = 1$ .

**Theorem 3.** *Let  $K \in \mathcal{K}^n$ ,  $\beta \in \mathcal{B}(\mathbb{R}^n)$ , and  $j, k, r, s \in \mathbb{N}_0$  with  $0 < j < k \leq n$ . Then,*

$$\int_{A(n,k)} \phi_j^{r,s,1}(K \cap E, \beta \cap E) \mu_k(dE) = \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} d_{n,j,k}^{s,1,0,m} Q^m \phi_{n-k+j}^{r,s-2m,1}(K, \beta),$$

where

$$d_{n,j,k}^{s,1,0,m} := \frac{1}{(4\pi)^m m!} \frac{\Gamma(\frac{n-k+j+1}{2})\Gamma(\frac{k+1}{2})}{\Gamma(\frac{n+1}{2})\Gamma(\frac{j+1}{2})} \frac{\Gamma(\frac{n-k+j}{2} + 1)}{\Gamma(\frac{n-k+j+s}{2} + 1)} \frac{\Gamma(\frac{j+s}{2} - m + 1)}{\Gamma(\frac{j}{2} + 1)} \frac{\Gamma(\frac{n-k}{2} + m)}{\Gamma(\frac{n-k}{2})}.$$

Next, we state the formula for  $l = 0$ .

**Theorem 4.** *Let  $K \in \mathcal{K}^n$ ,  $\beta \in \mathcal{B}(\mathbb{R}^n)$  and  $j, k, r, s \in \mathbb{N}_0$  with  $j < k \leq n$ . Then,*

$$\int_{A(n,k)} \phi_j^{r,s,0}(K \cap E, \beta \cap E) \mu_k(dE) = \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{i=0}^1 d_{n,j,k}^{s,0,i,m} Q^{m-i} \phi_{n-k+j}^{r,s-2m,i}(K, \beta),$$

where

$$d_{n,j,k}^{s,0,i,m} := \frac{1}{(4\pi)^m m!} \frac{\binom{m}{i}}{\pi^i} \frac{\Gamma(\frac{n-k+j+1}{2})\Gamma(\frac{k+1}{2})}{\Gamma(\frac{n+1}{2})\Gamma(\frac{j+1}{2})} \frac{\Gamma(\frac{n-k+j}{2} + 1)}{\Gamma(\frac{n-k+j+s}{2} + 1)} \frac{\Gamma(\frac{j+s}{2} - m + 1)}{\Gamma(\frac{j}{2} + 1)} \frac{\Gamma(\frac{n-k}{2} + m)}{\Gamma(\frac{n-k}{2})}.$$

In Theorem 4, we have  $d_{n,j,k}^{s,0,1,0} = 0$  so that in fact the undefined tensor  $Q^{-1}$  does not appear.

For the special case  $j = k - 1$ , we deduce two more Crofton formulae. The first concerns the generalized tensorial curvature measures  $\phi_{k-1}^{r,s,1}$ .

**Corollary 5.** *Let  $K \in \mathcal{K}^n$ ,  $\beta \in \mathcal{B}(\mathbb{R}^n)$ , and  $k, r, s \in \mathbb{N}_0$  with  $0 < k < n$ . Then,*

$$\int_{A(n,k)} \phi_{k-1}^{r,s,1}(K \cap E, \beta \cap E) \mu_k(dE) = \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \iota_{n,k}^{s,m} Q^m \phi_{n-1}^{r,s-2m,1}(K, \beta),$$

where

$$\iota_{n,k}^{s,m} := \frac{1}{(4\pi)^m m!} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{k+s+1}{2} - m)\Gamma(\frac{n-k}{2} + m)}{\Gamma(\frac{n+s+1}{2})\Gamma(\frac{k}{2})\Gamma(\frac{n-k}{2})}.$$

Due to the easily verified relation

$$(6) \quad \phi_{n-1}^{r,s-2m,1} = \frac{2\pi}{n-1} \left( Q\phi_{n-1}^{r,s-2m,0} - 2\pi(s-2m+2)\phi_{n-1}^{r,s-2m+2,0} \right),$$

Corollary 5 can be transformed in such a way that only the tensorial curvature measures  $\phi_{n-1}^{r,s-2m,0}$  are involved on the right-hand side of the preceding formula.

**Corollary 6.** *Let  $K \in \mathcal{K}^n$ ,  $\beta \in \mathcal{B}(\mathbb{R}^n)$ , and  $k, r, s \in \mathbb{N}_0$  with  $1 < k < n$ . Then,*

$$\int_{A(n,k)} \phi_{k-1}^{r,s,1}(K \cap E, \beta \cap E) \mu_k(dE) = \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor + 1} \lambda_{n,k}^{s,m} Q^m \phi_{n-1}^{r,s-2m+2,0}(K, \beta),$$

where

$$\lambda_{n,k}^{s,m} = \frac{\pi}{(n-1)(4\pi)^{m-1}m!} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{k+s+1}{2} - m)\Gamma(\frac{n-k}{2} + m - 1)}{\Gamma(\frac{n+s+1}{2})\Gamma(\frac{k}{2})\Gamma(\frac{n-k}{2})} \\ \times \left( 2m(\frac{k+s+1}{2} - m) - (s-2m+2)(\frac{n-k}{2} + m - 1) \right),$$

for  $m \in \{1, \dots, \lfloor \frac{s}{2} \rfloor\}$ , and

$$\lambda_{n,k}^{s,0} = -\frac{4\pi^2(s+2)}{n-1} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{k+s+1}{2})}{\Gamma(\frac{n+s+1}{2})\Gamma(\frac{k}{2})}, \\ \lambda_{n,k}^{s, \lfloor \frac{s}{2} \rfloor + 1} = \frac{2\pi}{(n-1)(4\pi)^{\lfloor \frac{s}{2} \rfloor} (\lfloor \frac{s}{2} \rfloor)!} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{k+s+1}{2} - \lfloor \frac{s}{2} \rfloor)\Gamma(\frac{n-k}{2} + \lfloor \frac{s}{2} \rfloor)}{\Gamma(\frac{n+s+1}{2})\Gamma(\frac{k}{2})\Gamma(\frac{n-k}{2})}.$$

The second special case concerns the tensorial curvature measures  $\phi_{k-1}^{r,s,0}$ . Though, the result has been derived in a different way as [14, Theorem 4.12], we state it and derive it again as a special case of the present more general approach.

**Corollary 7.** *Let  $K \in \mathcal{K}^n$ ,  $\beta \in \mathcal{B}(\mathbb{R}^n)$ , and  $k, r, s \in \mathbb{N}_0$  with  $1 < k < n$ . Then*

$$\int_{A(n,k)} \phi_{k-1}^{r,s,0}(K \cap E, \beta \cap E) \mu_k(dE) = \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \kappa_{n,k}^{s,m} Q^m \phi_{n-1}^{r,s-2m,0}(K, \beta),$$

where

$$\kappa_{n,k}^{s,m} := \frac{k-1}{n-1} \frac{1}{(4\pi)^m m!} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{k+s-1}{2} - m)\Gamma(\frac{n-k}{2} + m)}{\Gamma(\frac{n+s-1}{2})\Gamma(\frac{k}{2})\Gamma(\frac{n-k}{2})}$$

if  $m \neq \frac{s-1}{2}$ , and

$$\kappa_{n,k}^{s, \frac{s-1}{2}} := \frac{k(n+s-2)}{2(n-1)} \frac{1}{(4\pi)^{\frac{s-1}{2}} \frac{s-1}{2}!} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{n-k+s-1}{2})}{\Gamma(\frac{n+s+1}{2})\Gamma(\frac{n-k}{2})}.$$

Finally, we state the remaining case where  $k = 1$  (see also [14, Theorem 4.13]).



**Corollary 8.** *Let  $K \in \mathcal{K}^n$ ,  $\beta \in \mathcal{B}(\mathbb{R}^n)$ , and  $r, s \in \mathbb{N}_0$ . Then*

$$\begin{aligned} & \int_{A(n,1)} \phi_0^{r,s,0}(K \cap E, \beta \cap E) \mu_k(dE) \\ &= \frac{\Gamma(\frac{s}{2} - \lfloor \frac{s}{2} \rfloor + 1) \Gamma(\frac{n}{2}) \Gamma(\frac{n+1}{2} + \lfloor \frac{s}{2} \rfloor)}{\sqrt{\pi} (4\pi)^{\lfloor \frac{s}{2} \rfloor} \lfloor \frac{s}{2} \rfloor! \Gamma(\frac{n+1}{2}) \Gamma(\frac{n+s+1}{2})} Q^{\lfloor \frac{s}{2} \rfloor} \phi_{n-1}^{r,s-2\lfloor \frac{s}{2} \rfloor,0}(K, \beta). \end{aligned}$$

Comparing Corollary 7 and Corollary 8 to the corresponding results in [14], it should be observed that the normalization of the tensorial measures in [14] is different from the current normalization (although the measures are denoted in the same way).

#### 4. THE PROOFS OF THE MAIN RESULTS

**4.1. The kinematic formula for (generalized) tensorial curvature measures.** The proof of the Crofton formulae uses the connection to the corresponding (more general) kinematic formulae. For the classical scalar-valued curvature measures this connection is well known. For easier reference, we state the required kinematic formula, which has recently been proved in [15, Theorem 1]. To state the result, we write  $G_n$  for the rigid motion group of  $\mathbb{R}^n$  and denote by  $\mu$  the Haar measure on  $G_n$  with the usual normalization (see [14], [24, p. 586]).

**Theorem 9** (Kinematic formula [14]). *For  $P, P' \in \mathcal{P}^n$ ,  $\beta, \beta' \in \mathcal{B}(\mathbb{R}^n)$ ,  $j, l, r, s \in \mathbb{N}_0$  with  $j \leq n$ , and  $l = 0$  if  $j = 0$ ,*

$$\begin{aligned} & \int_{G_n} \phi_j^{r,s,l}(P \cap gP', \beta \cap g\beta') \mu(dg) \\ &= \sum_{p=j}^n \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{i=0}^m d_{n,j,n-p+j}^{s,l,i,m} Q^{m-i} \phi_p^{r,s-2m,l+i}(P, \beta) \phi_{n-p+j}(P', \beta'), \end{aligned}$$

where

$$\begin{aligned} d_{n,j,n-p+j}^{s,l,i,m} &:= \frac{(-1)^i}{(4\pi)^m m!} \frac{\binom{m}{i}}{\pi^i} \frac{(i+l-2)!}{(l-2)!} \frac{\Gamma(\frac{n-p+j+1}{2}) \Gamma(\frac{p+1}{2})}{\Gamma(\frac{n+1}{2}) \Gamma(\frac{j+1}{2})} \\ &\quad \times \frac{\Gamma(\frac{p}{2} + 1)}{\Gamma(\frac{p+s}{2} + 1)} \frac{\Gamma(\frac{j+s}{2} - m + 1)}{\Gamma(\frac{j}{2} + 1)} \frac{\Gamma(\frac{p-j}{2} + m)}{\Gamma(\frac{p-j}{2})}. \end{aligned}$$

In this formulation of Theorem 9, we changed the order of the coefficients slightly as compared to the original work (see [15, Theorem 1]), as we have  $d_{n,j,n-p+j}^{s,l,i,m} := c_{n,j,p}^{s,l,i,m}$ . This is done in order to shorten the representation of the Crofton formulae. Furthermore, since  $\phi_n^{r,s,l}$  vanishes for  $s \neq 0$  and the functionals  $Q^{\frac{s}{2}-i} \phi_n^{r,0,l+i}$ ,  $i \in \{0, \dots, \frac{s}{2}\}$ , can be combined, we can redefine

$$d_{n,j,j}^{s,l,i,m} := \mathbb{1}\{s \text{ even}, m = i = \frac{s}{2}\} \frac{1}{(2\pi)^s (\frac{s}{2})!} \frac{\Gamma(\frac{n-j+s}{2})}{\Gamma(\frac{n-j}{2})};$$

for further details see the remark after Theorem 1 in [15]. In particular, the ratio  $(i+l-2)!/(l-2)!$  is interpreted as a ratio of Gamma functions for  $l \in \{0, 1\}$  (see also the comments after Theorem 2).

**4.2. The proofs.** We prove both, Theorem 1 and Theorem 2, at once using the kinematic formula for generalized tensorial curvature measures deduced in [15] and restated in the last section as Theorem 9.

*Proof of Theorem 1 and Theorem 2.* Let  $P \in \mathcal{P}^n$  and  $\beta \in \mathcal{B}(\mathbb{R}^n)$ . First, we are going to prove the identity

$$(7) \quad J := \int_{A(n,k)} \phi_j^{r,s,l}(P \cap E, \beta) \mu_k(dE) = \int_{G_n} \phi_j^{r,s,l}(P \cap gE_k, \beta \cap g\alpha) \mu(dg)$$

for fixed  $E_k \in G(n, k)$  and  $\alpha \in \mathcal{B}(E_k)$  with  $\mathcal{H}^k(\alpha) = 1$ . This is shown in the following steps: By decomposing the measure  $\mu_k$  as usual, we obtain

$$J = \int_{\text{SO}(n)} \int_{E_k^\perp} \int_{\mathbb{R}^n} \mathbb{1}_\beta(x) \phi_j^{r,s,l}(P \cap \rho(E_k + t_1), dx) \mathcal{H}^{n-k}(dt_1) \nu(d\rho),$$

where  $\text{SO}(n)$  is the group of proper (orientation preserving) rotations of  $\mathbb{R}^n$  and  $\nu$  is the Haar probability measure on  $\text{SO}(n)$ .

For  $t_1 \in E_k^\perp$  and  $x \in \rho(E_k + t_1)$  we have

$$x \in \rho(\alpha + t_1 + t_2) \Leftrightarrow t_2 \in -\alpha + \rho^{-1}x - t_1,$$

for all  $t_2 \in E_k$ . Moreover,  $-\alpha + \rho^{-1}x - t_1 \subseteq E_k$ , since  $\alpha \subseteq E_k$  and  $x \in \rho(E_k + t_1)$  yields  $\rho^{-1}x - t_1 \in E_k$ . Thus, we get

$$\mathcal{H}^k(\{t_2 \in E_k : x \in \rho(\alpha + t_1 + t_2)\}) = \mathcal{H}^k(-\alpha + \rho^{-1}x - t_1) = \mathcal{H}^k(\alpha) = 1,$$

and hence we have

$$\begin{aligned} J &= \int_{\text{SO}(n)} \int_{E_k^\perp} \int_{\mathbb{R}^n} \mathbb{1}_\beta(x) \int_{E_k} \mathbb{1}\{x \in \rho(\alpha + t_1 + t_2)\} \mathcal{H}^k(dt_2) \\ &\quad \times \phi_j^{r,s,l}(P \cap \rho(E_k + t_1), dx) \mathcal{H}^{n-k}(dt_1) \nu(d\rho) \\ &= \int_{\text{SO}(n)} \int_{E_k^\perp} \int_{E_k} \int_{\mathbb{R}^n} \mathbb{1}_{\beta \cap \rho(\alpha + t_1 + t_2)}(x) \phi_j^{r,s,l}(P \cap \rho(E_k + t_1 + t_2), dx) \\ &\quad \times \mathcal{H}^k(dt_2) \mathcal{H}^{n-k}(dt_1) \nu(d\rho). \end{aligned}$$

Finally, Fubini's theorem yields

$$\begin{aligned} J &= \int_{\text{SO}(n)} \int_{\mathbb{R}^n} \phi_j^{r,s,l}(P \cap \rho(E_k + t), \beta \cap \rho(\alpha + t)) \mathcal{H}^n(dt) \nu(d\rho) \\ &= \int_{G_n} \phi_j^{r,s,l}(P \cap gE_k, \beta \cap g\alpha) \mu(dg), \end{aligned}$$

which concludes the proof of (7).

Let  $\alpha \in \mathcal{B}(\mathbb{R}^n)$  be compact with  $\alpha \subset E_k$  and  $\mathcal{H}^k(\alpha) = 1$ . Then choose  $P' \in \mathcal{P}^n$  with  $P' \subseteq E_k$  and  $\alpha \subseteq \text{relint } P'$ , such that the following holds, for all  $g \in G_n$ : If  $g^{-1}P \cap \alpha \neq \emptyset$ , then  $g^{-1}P \cap E_k \subset P'$ . Hence, if  $P \cap g\alpha \neq \emptyset$ , then  $P \cap gE_k = P \cap gP'$ . Thus we obtain

$$J = \int_{G_n} \phi_j^{r,s,l}(P \cap gP', \beta \cap g\alpha) \mu(dg),$$

and thus, by Theorem 9

$$J = \sum_{p=j}^n \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{i=0}^m d_{n,j,n-p+j}^{s,l,i,m} Q^{m-i} \phi_p^{r,s-2m,l+i}(P, \beta) \phi_{n-p+j}(P', \alpha).$$

Hence, if  $k = j$  we get

$$\begin{aligned} J &= \mathbb{1}\{s \text{ even}\} \frac{1}{(2\pi)^s \left(\frac{s}{2}\right)!} \frac{\Gamma\left(\frac{n-k+s}{2}\right)}{\Gamma\left(\frac{n-k}{2}\right)} \phi_n^{r,0,\frac{s}{2}+l}(P, \beta) \underbrace{\phi_k(P', \alpha)}_{=\mathcal{H}^k(\alpha)=1} \\ &= \mathbb{1}\{s \text{ even}\} \frac{1}{(2\pi)^s \frac{s}{2}!} \frac{\Gamma\left(\frac{n-k+s}{2}\right)}{\Gamma\left(\frac{n-k}{2}\right)} \phi_n^{r,0,\frac{s}{2}+l}(P, \beta), \end{aligned}$$

and for  $j < k$  we get

$$\begin{aligned} J &= \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{i=0}^m d_{n,j,k}^{s,l,i,m} Q^{m-i} \phi_{n-k+j}^{r,s-2m,l+i}(P, \beta) \underbrace{\phi_k(P', \alpha)}_{=\mathcal{H}^k(\alpha)=1} \\ &= \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \sum_{i=0}^m d_{n,j,k}^{s,l,i,m} Q^{m-i} \phi_{n-k+j}^{r,s-2m,l+i}(P, \beta), \end{aligned}$$

since  $\phi_q(P', \alpha) = 0$  for  $q \neq k$ . □

Next, we prove Corollary 5 and Corollary 6, which are derived from Theorem 3. The first follows immediately, whereas the second subsequently is obtained by an application of (6).

*Proof of Corollary 5 and Corollary 6.* We denote the integral, we are interested in, by  $I$ . Then Theorem 3 yields

$$I = \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \iota_{n,k}^{s,m} Q^m \phi_{n-1}^{r,s-2m,1}(K, \beta),$$

where

$$\iota_{n,k}^{s,m} := d_{n,k-1,k}^{s,1,0,m} = \frac{1}{(4\pi)^m m!} \frac{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{k+s+1}{2} - m\right) \Gamma\left(\frac{n-k}{2} + m\right)}{\Gamma\left(\frac{n+s+1}{2}\right) \Gamma\left(\frac{k}{2}\right) \Gamma\left(\frac{n-k}{2}\right)},$$

which already proves Corollary 5. Next, we conclude from (6)

$$\begin{aligned} I &= \frac{2\pi}{n-1} \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \iota_{n,k}^{s,m} Q^{m+1} \phi_{n-1}^{r,s-2m,0}(K, \beta) - 2\pi(s-2m+2) \iota_{n,k}^{s,m} Q^m \phi_{n-1}^{r,s-2m+2,0}(K, \beta) \\ &= \frac{2\pi}{n-1} \sum_{m=1}^{\lfloor \frac{s}{2} \rfloor + 1} \iota_{n,k}^{s,m-1} Q^m \phi_{n-1}^{r,s-2m+2,0}(K, \beta) \\ &\quad - \frac{2\pi}{n-1} \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} 2\pi(s-2m+2) \iota_{n,k}^{s,m} Q^m \phi_{n-1}^{r,s-2m+2,0}(K, \beta) \\ &= \sum_{m=1}^{\lfloor \frac{s}{2} \rfloor} \frac{2\pi}{n-1} \left( \iota_{n,k}^{s,m-1} - 2\pi(s-2m+2) \iota_{n,k}^{s,m} \right) Q^m \phi_{n-1}^{r,s-2m+2,0}(K, \beta) \\ &\quad + \frac{2\pi}{n-1} \iota_{n,k}^{s,\lfloor \frac{s}{2} \rfloor} Q^{\lfloor \frac{s}{2} \rfloor + 1} \phi_{n-1}^{r,s-2\lfloor \frac{s}{2} \rfloor,0}(K, \beta) - \frac{4\pi^2(s+2)}{n-1} \iota_{n,k}^{s,0} \phi_{n-1}^{r,s+2,0}(K, \beta). \end{aligned}$$

Denoting the coefficients by  $\lambda_{n,k}^{s,m}$ , we obtain for  $m \in \{1, \dots, \lfloor \frac{s}{2} \rfloor\}$

$$\lambda_{n,k}^{s,m} = \frac{\pi}{(n-1)(4\pi)^{m-1}m!} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{k+s+1}{2}-m)\Gamma(\frac{n-k}{2}+m-1)}{\Gamma(\frac{n+s+1}{2})\Gamma(\frac{k}{2})\Gamma(\frac{n-k}{2})} \\ \times \left(2m(\frac{k+s+1}{2}-m) - (s-2m+2)(\frac{n-k}{2}+m-1)\right),$$

and

$$\lambda_{n,k}^{s,0} = -\frac{4\pi^2(s+2)}{n-1} l_{n,k}^{s,0} = -\frac{4\pi^2(s+2)}{n-1} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{k+s+1}{2})}{\Gamma(\frac{n+s+1}{2})\Gamma(\frac{k}{2})}, \\ \lambda_{n,k}^{s, \lfloor \frac{s}{2} \rfloor + 1} = \frac{2\pi}{n-1} l_{n,k}^{s, \lfloor \frac{s}{2} \rfloor} = \frac{2\pi}{(n-1)(4\pi)^{\lfloor \frac{s}{2} \rfloor} (\lfloor \frac{s}{2} \rfloor)!} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{k+s+1}{2} - \lfloor \frac{s}{2} \rfloor)\Gamma(\frac{n-k}{2} + \lfloor \frac{s}{2} \rfloor)}{\Gamma(\frac{n+s+1}{2})\Gamma(\frac{k}{2})\Gamma(\frac{n-k}{2})},$$

where  $\lambda_{n,k}^{s,0}$  is defined according to the general definition, but  $\lambda_{n,k}^{s, \lfloor \frac{s}{2} \rfloor + 1}$  differs slightly for odd  $s$ .  $\square$

Finally, we prove Corollary 7 and Corollary 8, which are derived from Theorem 4.

*Proof of Corollary 7 and 8.* We denote the integral, we are interested in, by  $I$ . Theorem 4 yields

$$I = \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} d_{n,k-1,k}^{s,0,0,m} Q^m \phi_{n-1}^{r,s-2m,0}(K, \beta) + \sum_{m=1}^{\lfloor \frac{s}{2} \rfloor} d_{n,k-1,k}^{s,0,1,m} Q^{m-1} \phi_{n-1}^{r,s-2m,1}(K, \beta),$$

where

$$d_{n,k-1,k}^{s,0,i,m} := \frac{1}{4^m(m-i)!} \frac{1}{\pi^{i+m}} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n+s+1}{2})\Gamma(\frac{k}{2})\Gamma(\frac{n-k}{2})} \Gamma(\frac{k+s+1}{2}-m)\Gamma(\frac{n-k}{2}+m).$$

From (6) we obtain that

$$I = \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} d_{n,k-1,k}^{s,0,0,m} Q^m \phi_{n-1}^{r,s-2m,0}(K, \beta) + \frac{2\pi}{n-1} \sum_{m=1}^{\lfloor \frac{s}{2} \rfloor} d_{n,k-1,k}^{s,0,1,m} Q^m \phi_{n-1}^{r,s-2m,0}(K, \beta) \\ - \frac{4\pi^2}{n-1} \sum_{m=1}^{\lfloor \frac{s}{2} \rfloor} d_{n,k-1,k}^{s,0,1,m} (s-2m+2) Q^{m-1} \phi_{n-1}^{r,s-2m+2,0}(K, \beta).$$

As  $d_{n,k-1,k}^{s,0,1,0} = 0$ , we can rewrite

$$I = \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor} \left( d_{n,k-1,k}^{s,0,0,m} + \frac{2\pi}{n-1} d_{n,k-1,k}^{s,0,1,m} \right) Q^m \phi_{n-1}^{r,s-2m,0}(K, \beta) \\ - \frac{4\pi^2}{n-1} \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor - 1} d_{n,k-1,k}^{s,0,1,m+1} (s-2m) Q^m \phi_{n-1}^{r,s-2m,0}(K, \beta) \\ = \sum_{m=0}^{\lfloor \frac{s}{2} \rfloor - 1} \left( d_{n,k-1,k}^{s,0,0,m} + \frac{2\pi}{n-1} d_{n,k-1,k}^{s,0,1,m} - \frac{4\pi^2(s-2m)}{n-1} d_{n,k-1,k}^{s,0,1,m+1} \right) Q^m \phi_{n-1}^{r,s-2m,0}(K, \beta) \\ + \left( d_{n,k-1,k}^{s,0,0, \lfloor \frac{s}{2} \rfloor} + \frac{2\pi}{n-1} d_{n,k-1,k}^{s,0,1, \lfloor \frac{s}{2} \rfloor} \right) Q^{\lfloor \frac{s}{2} \rfloor} \phi_{n-1}^{r,s-2\lfloor \frac{s}{2} \rfloor,0}(K, \beta).$$

Denoting the corresponding coefficients of the summand  $Q^m \phi_{n-1}^{r,s-2m,0}(K, \beta)$  by  $\kappa_{n,k}^{s,m}$ , we obtain

$$\begin{aligned}
 \kappa_{n,k}^{s,m} &= \left(1 + \frac{2m}{n-1}\right) \frac{1}{(4\pi)^m m!} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{k+s+1}{2}-m)\Gamma(\frac{n-k}{2}+m)}{\Gamma(\frac{n+s+1}{2})\Gamma(\frac{k}{2})\Gamma(\frac{n-k}{2})} \\
 &\quad - \frac{s-2m}{n-1} \frac{1}{(4\pi)^m m!} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{k+s-1}{2}-m)\Gamma(\frac{n-k}{2}+m+1)}{\Gamma(\frac{n+s+1}{2})\Gamma(\frac{k}{2})\Gamma(\frac{n-k}{2})} \\
 &= \frac{1}{(4\pi)^m m!} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{k+s-1}{2}-m)\Gamma(\frac{n-k}{2}+m)}{\Gamma(\frac{n+s+1}{2})\Gamma(\frac{k}{2})\Gamma(\frac{n-k}{2})} \\
 &\quad \times \underbrace{\left(\frac{n+2m-1}{n-1}(\frac{k+s-1}{2}-m) - \frac{s-2m}{n-1}(\frac{n-k}{2}+m)\right)}_{=\frac{k-1}{n-1}\frac{n+s-1}{2}} \\
 (8) \quad &= \frac{k-1}{n-1} \frac{1}{(4\pi)^m m!} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{k+s-1}{2}-m)\Gamma(\frac{n-k}{2}+m)}{\Gamma(\frac{n+s-1}{2})\Gamma(\frac{k}{2})\Gamma(\frac{n-k}{2})},
 \end{aligned}$$

for  $m \in \{0, \dots, \lfloor \frac{s}{2} \rfloor - 1\}$ . For  $k = 1$ , we immediately get  $\kappa_{n,1}^{s,m} = 0$  in these cases. Furthermore, we have

$$\kappa_{n,k}^{s, \lfloor \frac{s}{2} \rfloor} = \left(1 + \frac{2\lfloor \frac{s}{2} \rfloor}{n-1}\right) \frac{1}{(4\pi)^{\lfloor \frac{s}{2} \rfloor} \lfloor \frac{s}{2} \rfloor!} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{k+s+1}{2}-\lfloor \frac{s}{2} \rfloor)\Gamma(\frac{n-k}{2}+\lfloor \frac{s}{2} \rfloor)}{\Gamma(\frac{n+s+1}{2})\Gamma(\frac{k}{2})\Gamma(\frac{n-k}{2})}.$$

If  $s$  is even and  $k > 1$ , this coincides with (8) for  $m = \frac{s}{2}$ . If  $s$  is odd, we have

$$\kappa_{n,k}^{s, \frac{s-1}{2}} = \frac{k(n+s-2)}{2(n-1)} \frac{1}{(4\pi)^{\frac{s-1}{2}} \frac{s-1}{2}!} \frac{\Gamma(\frac{n}{2})\Gamma(\frac{n-k+s-1}{2})}{\Gamma(\frac{n+s+1}{2})\Gamma(\frac{n-k}{2})}$$

and thus the assertion of Corollary 7. For  $k = 1$ , we obtain

$$\kappa_{n,1}^{s, \lfloor \frac{s}{2} \rfloor} = \frac{\Gamma(\frac{s}{2} - \lfloor \frac{s}{2} \rfloor + 1) \Gamma(\frac{n}{2})\Gamma(\frac{n+1}{2} + \lfloor \frac{s}{2} \rfloor)}{\sqrt{\pi}(4\pi)^{\lfloor \frac{s}{2} \rfloor} \lfloor \frac{s}{2} \rfloor! \Gamma(\frac{n+1}{2})\Gamma(\frac{n+s+1}{2})}$$

and thus the assertion of Corollary 8.  $\square$

## 5. LINEAR INDEPENDENCE OF THE TENSORIAL CURVATURE MEASURES

In this section, we show that the tensorial curvature measures are linearly independent. The proof of this result follows the argument for Theorem 3.1 in [9].

**Theorem 10.** *For  $p \in \mathbb{N}_0$ , the (generalized) tensorial curvature measures  $Q^m \phi_j^{r,s,l}$  with  $m, r, s, l \in \mathbb{N}_0$  and  $j \in \{0, \dots, n\}$ , where  $2m+2l+r+s = p$ , but with  $l = 0$  if  $j \in \{0, n-1\}$  and with  $s = l = 0$  if  $j = n$ , are linearly independent.*

*Proof.* Suppose that

$$(9) \quad \sum_{\substack{j,m,r,s,l \\ 2m+2l+r+s=p}} a_{j,m,r,s,l}^{(0)} Q^m \phi_j^{r,s,l} = 0$$

holds for  $a_{j,m,r,s,l}^{(0)} \in \mathbb{R}$ , where  $a_{0,m,r,s,l}^{(0)} = a_{n-1,m,r,s,l}^{(0)} = 0$  if  $l \neq 0$  and  $a_{n,m,r,s,l}^{(0)} = 0$  if  $s \neq 0$  or  $l \neq 0$ . Throughout the proof we will eventually replace the constants  $a_{j,l,m,r,s}^{(i)}$  by new

constants  $a_{j,l,m,r,s}^{(i+1)}$ ,  $i \in \mathbb{N}_0$ , without keeping track of the precise relations, since it will be sufficient to know that  $a_{j,m,r,s,l}^{(i)} = 0$  if and only if  $a_{j,m,r,s,l}^{(i+1)} = 0$  for all  $i \in \mathbb{N}_0$ .

For a fixed  $j \in \{0, \dots, n\}$ , let  $P \in \mathcal{P}^n$  with  $\text{int } P \neq \emptyset$ ,  $F \in \mathcal{F}_j(P)$ , and  $\beta \in \mathcal{B}(\text{relint } F)$ . Then, for  $j \in \{0, \dots, n-1\}$  we obtain for the tensorial curvature measures

$$\begin{aligned} \phi_j^{r,s,l}(P, \beta) &= c_{n,j,r,s,l} \sum_{G \in \mathcal{F}_j(P)} Q(G)^l \int_{G \cap \beta} x^r \mathcal{H}^j(dx) \int_{N(P,G) \cap \mathbb{S}^{n-1}} u^s \mathcal{H}^{n-j-1}(du) \\ &= c_{n,j,r,s,l} Q(F)^l \int_{\beta} x^r \mathcal{H}^j(dx) \int_{N(P,F) \cap \mathbb{S}^{n-1}} u^s \mathcal{H}^{n-j-1}(du), \end{aligned}$$

and  $\phi_k^{r,s,l}(P, \beta) = 0$  for  $k \neq j$ , where  $c_{n,j,r,s,l} > 0$  is a constant. Moreover, we have

$$\phi_n^{r,0,0}(P, \beta) = \frac{1}{r!} \int_{\beta} x^r \mathcal{H}^n(dx).$$

Hence, from (9) it follows that

$$\sum_{\substack{m,r,s,l \\ 2m+2l+r+s=p}} a_{j,m,r,s,l}^{(1)} Q^m Q(F)^l \int_{\beta} x^r \mathcal{H}^j(dx) \int_{N(P,F) \cap \mathbb{S}^{n-1}} u^s \mathcal{H}^{n-j-1}(du) = 0,$$

where for  $j = n$  the spherical integral is omitted (also in the following).

We may assume that  $\int_{\beta} x^r \mathcal{H}^j(dx) \neq 0$  (otherwise, we consider a translate of  $P$  and  $\beta$ ). If we consider the above calculations with multiples of  $P$  and  $\beta$ , a comparison of the degrees of homogeneity yields, for every  $r \in \mathbb{N}_0$ , that

$$\sum_{\substack{m,s,l \\ 2m+2l+s=p-r}} a_{j,m,r,s,l}^{(1)} Q^m Q(F)^l \int_{\beta} x^r \mathcal{H}^j(dx) \int_{N(P,F) \cap \mathbb{S}^{n-1}} u^s \mathcal{H}^{n-j-1}(du) = 0.$$

Hence, due to the lack of zero divisors in the tensor algebra  $\mathbb{T}$ , we obtain

$$(10) \quad \sum_{\substack{m,s,l \\ 2m+2l+s=p-r}} a_{j,m,r,s,l}^{(1)} Q^m Q(F)^l \int_{N(P,F) \cap \mathbb{S}^{n-1}} u^s \mathcal{H}^{n-j-1}(du) = 0.$$

This shows that for  $j = n$  (where the spherical integral is omitted) we have  $a_{n,m,r,s,l}^{(1)} = 0$  also for  $s = l = 0$ . Hence, in the following we may assume that  $j < n$ .

Let  $L \in \mathcal{G}(j)$ ,  $j < n$ , and  $u_0 \in L^\perp \cap \mathbb{S}^{n-1}$ . We claim that there is an  $(n-l)$ -dimensional polyhedral convex cone  $C \subset L^\perp$  such that

$$(11) \quad \int_{C \cap \mathbb{S}^{n-1}} u^s \mathcal{H}^{n-1-j}(du) = \lambda u_0^s,$$

for some constant  $\lambda > 0$ . This assertion is clear for  $j = n-1$ . For  $j \leq n-2$  let  $u_0, u_1, \dots, u_{n-1-j}$  be an orthonormal basis of  $L^\perp$ . Define the (peaked) polyhedral cone  $C := \text{pos}\{u_0 \pm u_1, \dots, u_0 \pm u_{n-1-j}\} \subset L^\perp$ . It is easy to check that  $\langle v, u_0 \rangle \geq [1 + 2(n-j-1)]^{-1}$  for all  $v \in C \cap \mathbb{S}^{n-1}$ . The reflection symmetry of  $C$  implies that (11) is satisfied.

Let  $C^\circ$  denote the polar cone of  $C$ . Then  $P := C^\circ \cap [-1, 1]^n \in \mathcal{P}^n$  and  $F := L \cap [-1, 1]^n \in \mathcal{F}_j(P)$  satisfy  $N(P, F) = N(P, 0) = C$ . This shows that for any subspace  $L \in \mathcal{G}(j)$ ,  $j < n$ , and any  $u_0 \in L^\perp \cap \mathbb{S}^{n-1}$ , there is a convex polytope  $P$  and a face  $F \in \mathcal{F}_j(P)$  such that  $L$  is the direction space of  $F$  and

$$\int_{N(P,F) \cap \mathbb{S}^{n-1}} u^s \mathcal{H}^{n-1-j}(du) = \lambda u_0^s$$

with some  $\lambda > 0$ .

Therefore, (10) implies that

$$\sum_{\substack{m,s,l \\ 2m+2l+s=p-r}} a_{j,m,r,s,l}^{(2)} Q^m Q(F)^l u^s = 0$$

for any  $u \in L^\perp \cap \mathbb{S}^{n-1}$ .

The rest of the proof follows similarly to the proof of [9, Theorem 3.1].

□

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KARLSRUHE INSTITUTE OF TECHNOLOGY (KIT), DEPARTMENT OF MATHEMATICS, D-76128 KARLSRUHE, GERMANY

*E-mail address:* daniel.hug@kit.edu

KARLSRUHE INSTITUTE OF TECHNOLOGY (KIT), DEPARTMENT OF MATHEMATICS, D-76128 KARLSRUHE, GERMANY

*E-mail address:* jan.weis@kit.edu